THE SEARCH FOR INDECOMPOSABLE MODULES

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DEFINITIONS AND STUFF

Let $R = \mathbb{k}[x, y, z]/(x^2 - y^2, x^2 - z^2, xy, xz, yz)$.

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As a $\Bbbk$-vector space,

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 & & & & \langle \overline{x^2} \rangle & & & & \text{degree 2, } \text{“}R_2\text{”} \\
 & & & & \oplus & & & & \\
R & \cong & \langle \bar{x} \rangle & \oplus & \langle \bar{y} \rangle & \oplus & \langle \bar{z} \rangle & & \text{degree 1, } \text{“}R_1\text{”} \\
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$$

The (finite) sequence $H_R = (1, 3, 1)$ is called the **Hilbert function of** R.

MY FAVORITE EXAMPLE

The R-module $M = R/(\bar{z})$ has Hilbert function $H_M = (1, 2, 0) = (1, 2)$:

$$M_2 = \langle \overline{x^2} \rangle = \langle \overline{z^2} \rangle = 0$$
$$M_1 = \langle \bar{x} \rangle \oplus \langle \bar{y} \rangle,$$
$$M_0 = \langle \bar{1} \rangle.$$

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Consider

$$0 \to (\bar{z}) \to R \to R/(\bar{z}) \to 0.$$

We can calculate $H_{(\bar{z})}$ as follows:

$$\begin{aligned} H_{(\bar{z})} &= H_R - H_{R/(\bar{z})} \\ &= (1,3,1) - (1,2,0) \\ &= (0,1,1). \end{aligned}$$

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Note that $R \oplus (\bar{z})$ has Hilbert function $(1,4,2)$.

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Fact 2. Suppose

$$0 \to I \to R \to F^1 \to F^2 \to \cdots \to F^n \to M \to 0$$

is an exact sequence of R-modules, where F^i is a free R-module for each i. Then M is called the n-th **cosyzygy of** I, $\mathrm{Cosyz}^n(I)$, and M is indecomposable.

BACKGROUND

SETTING UP THE QUESTION

Recall, $R = \mathbb{k}[x, y, z]/(x^2 - y^2, x^2 - z^2, xy, xz, yz)$.

More generally, $R = \Bbbk[x_1, \ldots, x_e]/(x_i^2 - x_j^2, x_i x_j \mid 1 \leq i < j \leq e)$ is called a **Short Gorenstein Ring of embedding dimension** e.

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Prop (Avramov-Iyengar-Şega). Let R be a S.G.R. with $e \geq 3$. If M is a Koszul R-module, then the Hilbert function $H_M = (p, q)$ where

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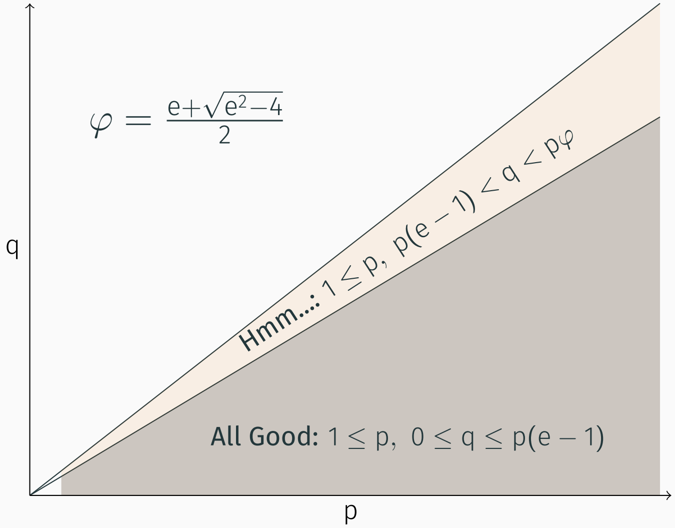
Furthermore, given

$$1 \leq p \quad \text{and} \quad 0 \leq q \leq p(e-1),$$

there exists a Koszul R-module M where $H_M = (p, q)$.

BUT HERE'S HOW TO REALLY THINK ABOUT IT



When $e = 3$,

$$\begin{aligned}\varphi &= \frac{e + \sqrt{e^2 - 4}}{2} \\ &= \frac{3 + \sqrt{5}}{2} \\ &= 1 + \frac{1 + \sqrt{5}}{2}\end{aligned}$$

THE GOLDEN TOUCH

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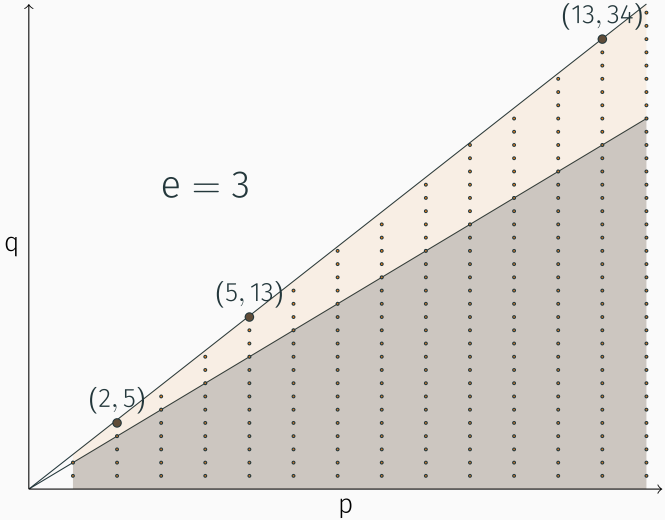
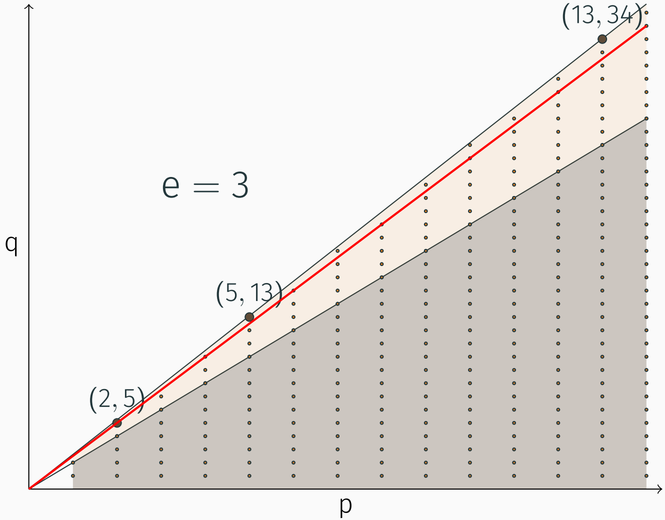
Geometrically, that means it's enough to find modules with the Hilbert functions

(1,2), (2,5), (5,13), (13,34), ...

ONCE MORE IN PICTURES

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e = 3
q
(5, 13)
(2, 5)
p

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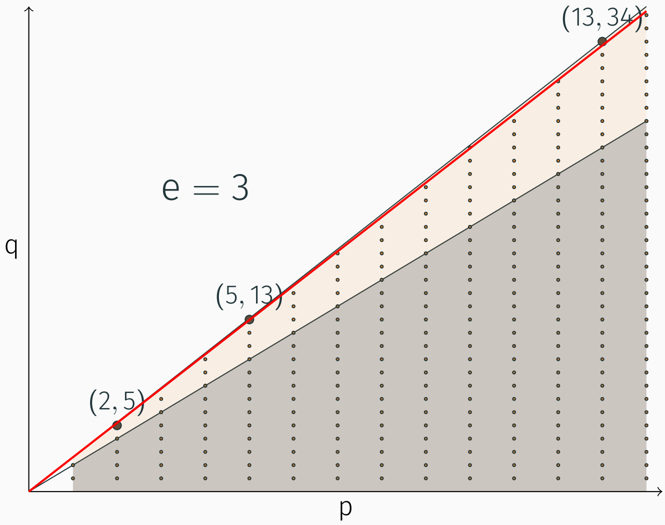

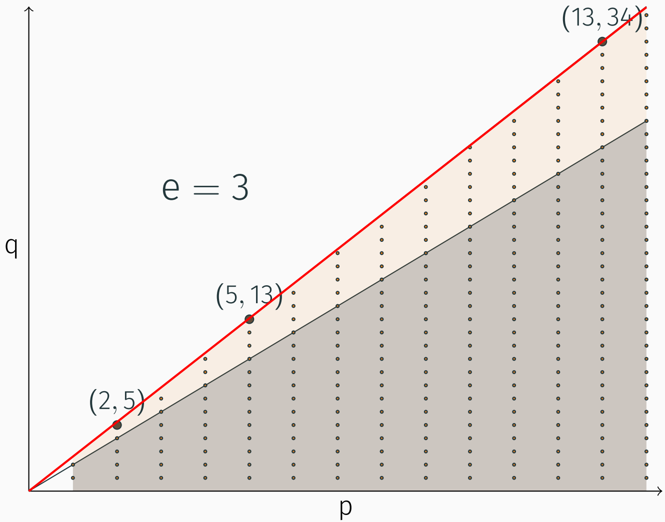
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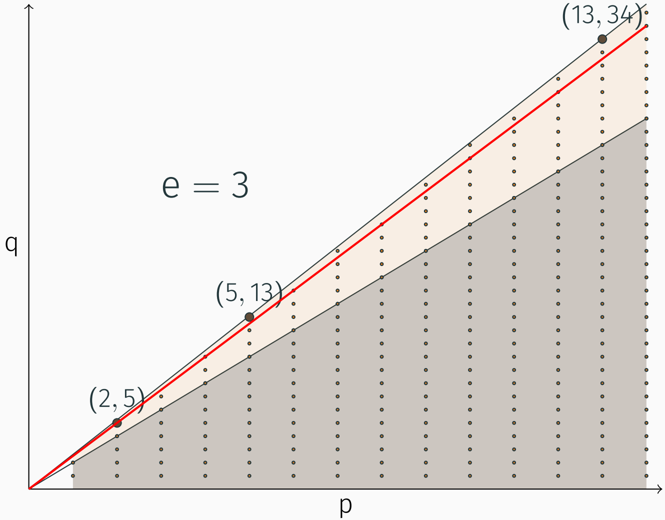
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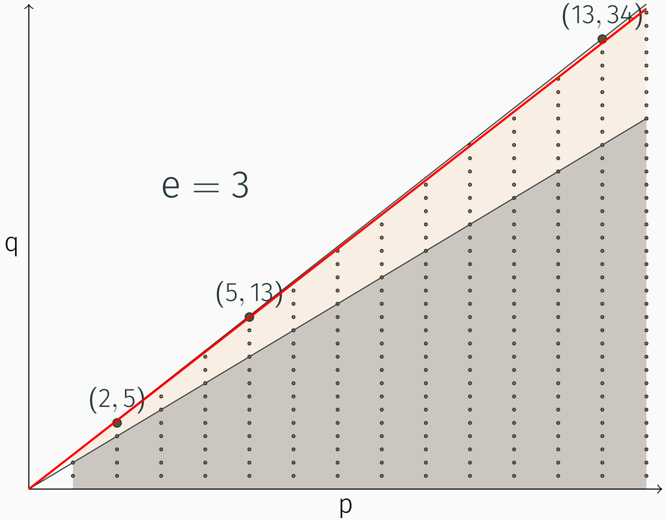

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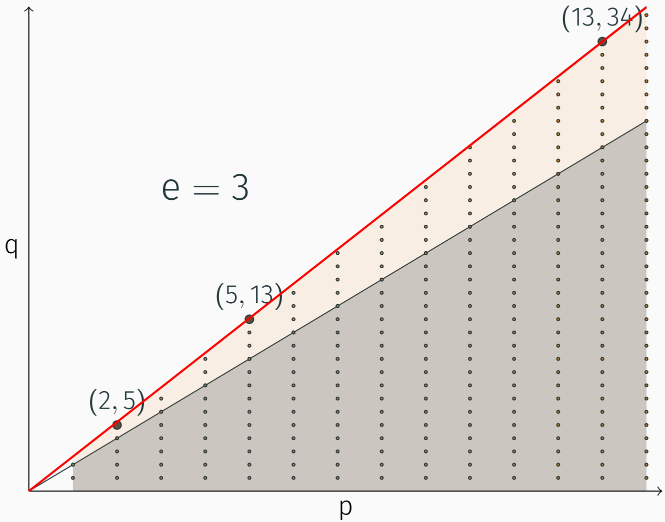

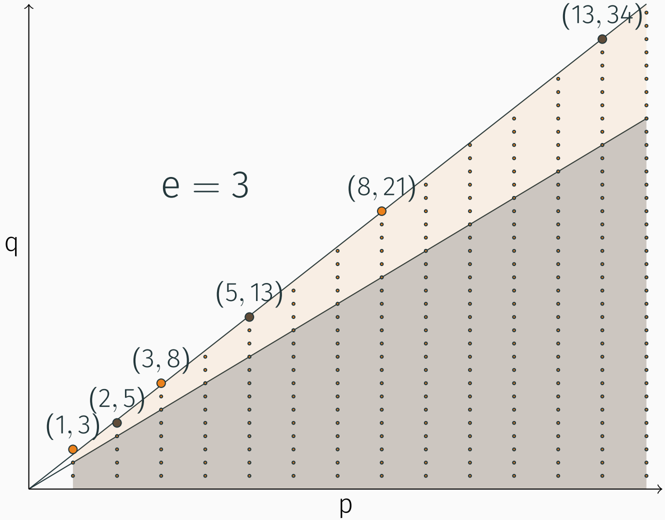
(13, 34)
e = 3
(8, 21)
q
(5, 13)
(3, 8)
(2, 5)
(1, 3)
p

Recap:

Fact 1. Hilbert function - additive on S.E.S.

Fact 2. There's an S.E.S.

$$0 \to \mathrm{Cosyz}_{n-1}(I) \to F^n \to \mathrm{Cosyz}_n(I) \to 0$$

(and $\mathrm{Cosyz}_n(I)$ is indecomposable).

CONSTRUCTING THE MODULES

Consider $I = (\bar{z})$. We know $H_{\mathrm{Cosyz}_1(I)} = H_{R/I} = (1, 2)$.

Then the Hilbert function of $\mathrm{Cosyz}_2(I)$ can be found from the exact sequence

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Repeating this trick,

$$H_{\mathrm{Cosyz}_3(I)} = (5, 15, 5) - (0, 2, 5) = (5, 13).$$

THEOREM (AVRAMOV-GIBBONS-WIEGAND)

Let R be a S.G.R. with $e \geq 3$. Then there exists a Koszul R-module M with Hilbert function (p, q) if and only if

$$1 \leq p \quad \text{and} \quad 0 \leq q \leq p\frac{e + \sqrt{e^2 - 4}}{2}.$$

QUESTIONS?

Definition. An R-module M is said to be **Koszul** provided M is generated in degree 0, has no nonzero free summand, and has a linear free resolution.

In our setting, if $H_M = (p, q)$, then:

$$\beta(M) = \begin{bmatrix} & & \vdots & & \\ 0 & 0 & \cdots & 0 & \cdots \\ p & ep - q & \cdots & (e\beta_{n-1} - \beta_{n-2}) & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ & & \vdots & & \end{bmatrix}.$$

THOSE OTHER ORDERED PAIRS? (1,3), (3,8), ...

The only indecomposable non-Koszul R-modules have the form $\mathrm{Cosyz}_n(\overline{x_1}, \ldots, \overline{x_e})$, and their Hilbert functions use the odd Fibonacci numbers.

SELECTED REFERENCES

[1] Luchezar L. Avramov, Srikanth B. Iyengar, and Liana M. Şega, Free resolutions over short local rings, J. Lond. Math. Soc. (2) **78** (2008), no. 2, 459–476, DOI 10.1112/jlms/jdn027. MR2439635 (2009h:13011)

[2] ______, Short Koszul modules, J. Commut. Algebra **2** (2010), no. 3, 249–279, DOI 10.1216/JCA-2010-2-3-249. MR2728144 (2012a:13025)

[3] Courtney Gibbons, Decompositions of Betti diagrams, ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–The University of Nebraska - Lincoln. MR3153511

[4] G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, 6th ed., Oxford University Press, Oxford, 2008. Revised by D. R. Heath-Brown and J. H. Silverman; With a foreword by Andrew Wiles. MR2445243 (2009i:11001)